Signal Processing 29 (1992) 335–347
Elsevier


# A new WLS Chebyshev approximation method for the design of FIR digital filters with arbitrary complex frequency response

Chong-Yung Chi and Shyu-Li Chiou
*Department of Electrical Engineering, National Tsing Hua University, Hsinchu, Taiwan, ROC*

Received 18 November 1991
Revised 31 March 1992

**Abstract.** In this paper, motivated by Chi–Kou's weighted least-squares (WLS) Chebyshev approximation method for the design of FIR filters with linear phase, we propose a new self-initiated iterative WLS Chebyshev approximation method for the design of FIR digital filters with arbitrary complex frequency response and real filter coefficients. The proposed method not only inherits all the advantages of Chi–Kou's method but also is computationally more efficient than Chi–Kou's method for the case of linear phase FIR filter design. On the other hand, contrast to Chit–Mason's time-domain least mean square (LMS) approximation methods, the proposed method is a frequency-domain WLS method based on similar philosophies. Several design examples are provided to demonstrate the good performance of the proposed method.

**Zusammenfassung.** Diese Arbeit wurde angeregt durch Chi–Kou's gewichtete Least-Squares (WLS) Tschebyscheff-Approximation zum Entwurf von FIR-Filtern mit linearer Phase. Es wird eine selbst-initialisierende iterative WLS-Tschebyscheff-Approximationsmethode zum Entwurf von FIR-Filtern mit beliebigem komplexem Frequenzgang und reellen Filterkoeffizienten vorgeschlagen. Dieses Verfahren übernimmt nicht nur die Vorteile der Methode von Chi–Kou, sondern ist zudem rechentechnisch effizienter für den Fall des Entwurfs linearphasiger Filter. Auf der anderen Seite ist die vorgeschlagene Methode im Gegensatz zum Zeitbereichsverfahren von Chit–Mason (Least-Mean-Square, LMS) eine WLS-Frequenzbereichs-Methode, die auf entsprechenden Grundlagen basiert. Es werden verschiedene Beispiele wiedergegeben, durch die die Leistungsfähigkeit des vorgeschlagenen Verfahrens demonstriert wird.

**Résumé.** Dans cet article motivé par la méthode d'approximation de Tchebycheff avec des moindres carrés pondérés, proposés par Chi–Kou, pour l'élaboration des filtres RIF à phase linéaire, nous proposons une approximation de Tchebycheff itérative auto-initiée pour l'élaboration des filtres RIF numériques avec une réponse fréquentielle complexe arbitraire et des coefficients réels. La méthode proposée hérite non seulement de tous les avantages de la méthode de Chi–Kou, mais est également plus efficace du point de vue de calcul que la méthode de Chi–Kou, pour le cas des filtres à phase linéaire RIF. D'un autre côté, en contraste avec la méthode d'approximation de moindres carrés du domaine temporel de Chit–Mason, la méthode proposée est une méthode de moindres carrés pondérés dans le domaine fréquentiel basé sur des philosophies similaires. Plusieurs exemples d'élaboration sont fournis pour démontrer les bonnes performances de la méthode proposée.

**Keywords.** Weighted least-squares (WLS) estimator; complex Chebyshev approximation; equiripple FIR digital filter; absolute approximation error.

## 1. Introduction

Finite duration impulse response (FIR) digital filters have been widely used in various signal processing application areas such as speech processing, image processing, communications, seismology, radar and sonar because they are inherently stable and linear phase can be easily attained. The windowing method and the optimum approximation method are the two major FIR filter design methods. The latter outperforms the former in that the

*Correspondence to*: C.-Y. Chi, Department of Electrical Engineering, National Tsing Hua University, Hsinchu, Taiwan, ROC.

required order of the filter is smaller for the same specifications, and in that different approximation errors in different frequency bands can be easily accommodated. Rivlin and Shapiro [7, 19, 20] have shown that the best Chebyshev approximation is also a best weighted least-squares (WLS) approximation with a suitable weighting function. Lawson [17, 18] proposed a fundamental algorithm, which is an iterative procedure for seeking a suitable weighting function and the extremal set, for computing the best Chebyshev approximation ($l_\infty$ approximation) by means of a weighted $l_p$ approximation ($p<\infty$). It has been reported in [7, 17, 18] that this algorithm converges very slowly and some modifications were suggested to accelerate the algorithm. However, it has been observed by Lawson, Rice and Usow [18] that these modifications sometimes lead to divergence. Parks and McClellan [11, 14] proposed an iterative algorithm for linear phase FIR filter design, which is a linear weighted Chebyshev approximation, allowing the exact specification of the cutoff frequencies. The well-known Parks–McClellan algorithm is flexible and computationally efficient, although it needs a suitable initial set of extremal frequencies. Recently, Chi and Kou [3, 4] proposed an efficient WLS Chebyshev approximation method for the design of linear phase FIR filters which performs as well as the well-known Parks–McClellan algorithm [11, 14] with some extra advantages. For instance, the former does not need the initial guess for a suitable set of extremal frequencies or filter coefficients; it is applicable for any type linear phase FIR filters with no need of nontrivial modifications required by the latter. However, these two optimum approximation methods are not applicable in the design of FIR filters with arbitrary complex frequency response.

As to complex Chebyshev approximation problems, a number of design techniques have been reported in the open literature. Steiglitz [22] proposed a method for designing FIR all-pass phase equalizers by linear programming. He showed that the solution of the linear programming is optimum to first order. Chen and Parks [2] approached a complex approximation problem by converting it into a nearly equivalent, real approximation problem which can be solved by a standard linear programming algorithm. Preuss [16] formulated a complex approximation problem as an equalization problem and then used a generalization of the Remez exchange algorithm to solve it. Schulist [21] modified the Preuss algorithm to accelerate it. Besides, he showed that if nonlinear phase filters have to be designed, the substitution of the Newton interpolation formula by a Gaussian relaxation algorithm yields good results, but sometimes it does not converge to the optimum solution for the case of linear phase filters. Pei and Shyu [15] converted the complex Chebyshev approximation into two real Chebyshev approximations to each of which the Remez exchange algorithm can be applied. Chit and Mason [5, 10] proposed time-domain least mean square (LMS) approximation methods which iteratively update the filter input as well as the filter coefficients, where the filter input consists of sinusoidal signals of all sample frequencies of interest, such that the approximation error is equiripple. Other complex approximation methods can be found in such as [1, 6, 8, 9, 23–25].

In this paper, we propose a new WLS Chebyshev approximation method motivated by the previously mentioned Chi–Kou's WLS Chebyshev approximation method [3, 4] for the design of FIR digital filters with arbitrary complex frequency response. The proposed method not only inherits all the advantages of Chi–Kou's method but also is computationally more efficient than Chi–Kou's method for the linear phase case. On the other hand, the proposed approximation method can be viewed as a frequency-domain counterpart to Chit–Mason's approximation LMS methods [5, 10] based on similar philosophies.

In Section 2, we present the new WLS Chebyshev approximation method. In Section 3, five examples including two lowpass filters, a full-band differentiator, a chirp all-pass phase equalizer and a sine-delay all-pass phase equalizer are provided to demonstrate the good performance of the proposed WLS Chebyshev approximation method.

Finally, we provide a discussion and draw some conclusions.

## 2. The new WLS Chebyshev approximation method

Assume that the desired complex frequency response $H_d(f)$ is conjugate-symmetric, i.e., $H_d(f)=H_d^*(-f)$, and that the domain $B_{NT}\subset[0, 0.5]$ over which $H_d(f)$ is well defined includes $p$ disjoint nontransition bands as follows:

$$B_{NT}=B_1\cup B_2\cup\cdots\cup B_p,$$

where

$$B_m=\{f\,|\,f_{m1}\leqslant f\leqslant f_{m2}\},\quad m=1,2,\ldots,p,$$

$f_{m1}$ and $f_{m2}$ denote the specified cutoff frequencies in the $m$th frequency band $B_m$. Then the union of the transition bands, denoted $B_{TS}$, is given by

$$B_{TS}=\{f\,|\,0\leqslant f\leqslant\tfrac{1}{2}, f\notin B_{NT}\}.$$

Assume that the filter to be designed is an $(M-1)$th-order FIR filter with real filter coefficients $h(n)$. The frequency response is then

$$\begin{aligned}H(f)&=\sum_{n=0}^{M-1}h(n)\,\mathrm{e}^{-\mathrm{j}2\pi fn}\\&=\sum_{n=0}^{M-1}h(n)\cos(2\pi fn)\\&\quad-\mathrm{j}\sum_{n=0}^{M-1}h(n)\sin(2\pi fn).\end{aligned}\tag{1}$$

We define the complex approximation error $E(f)$ as

$$\begin{aligned}E(f)&=H_d(f)-H(f)\\&=E_r(f)+\mathrm{j}E_i(f),\quad f\in B_{NT},\end{aligned}\tag{2}$$

where $E_r(f)$ and $E_i(f)$ are the real part and the imaginary part of $E(f)$, respectively. Let $W_e(f)$, $f\in B_{NT}$, be a piecewise-constant function associated with the desired relative approximation error ratio among $p$ frequency bands, defined as

$$W_e(f)=\rho_m,\quad \text{if } f\in B_m,\tag{3}$$

where $\rho_1>0$, $\rho_2>0,\ldots,\rho_p>0$, $\max\{\rho_1,\rho_2,\ldots,\rho_p\}=1$, and the ratio $(1/\rho_1):(1/\rho_2):\cdots:(1/\rho_p)$ denotes the desired relative approximation error ratio among $B_1, B_2, \ldots, B_p$. Our object is to find a set of filter coefficients $h(n)$ by the WLS estimator such that $H(f)$ is equiripple with $\delta_1:\delta_2:\cdots:\delta_p=(1/\rho_1):(1/\rho_2):\cdots:(1/\rho_p)$, where $\delta_m$ is the maximum approximation error in $B_m$. Next, let us present the WLS estimator on which the new approximation design method is based.

For notational simplicity, let $H_d(k)$, $W_e(k)$, $E(k)$, $E_r(k)$ and $E_i(k)$ also denote $H_d(f=k/2N)$, $W_e(f=k/2N)$, $E(f=k/2N)$, $E_r(f=k/2N)$ and $E_i(f=k/2N)$, respectively, where $N$ is the total number of uniform samples in the interval [0, 0.5]. Thus, by (1) and (2), we can express $E_r(k)$ and $E_i(k)$ for $k=0,1,\ldots,N-1$, in the following linear vector form:

$$\begin{bmatrix}\boldsymbol{E}_r\\ \boldsymbol{E}_i\end{bmatrix}=\begin{bmatrix}\mathrm{Re}(\boldsymbol{H}_d)\\ \mathrm{Im}(\boldsymbol{H}_d)\end{bmatrix}-\begin{bmatrix}D_1\\ D_2\end{bmatrix}\boldsymbol{h},\tag{4}$$

where $\boldsymbol{E}_r$ and $\boldsymbol{E}_i$ denote the real part and the imaginary part of $\boldsymbol{E}$, respectively, $\mathrm{Re}(\boldsymbol{H}_d)$ and $\mathrm{Im}(\boldsymbol{H}_d)$ denote the real part and the imaginary part of $\boldsymbol{H}_d$, respectively,

$$\boldsymbol{h}=[h(0), h(1), \ldots, h(M-1)]',$$

$$\boldsymbol{E}=[E(0), E(1), \ldots, E(N-1)]',$$

$$\boldsymbol{H}_d=[H_d(0), H_d(1), \ldots, H_d(N-1)]',$$

and $D_1$ as well as $D_2$ are $N\times M$ matrices with the $(i,j)$th element $[D_1]_{ij}=\cos((i-1)(j-1)\pi/N)$ and $[D_2]_{ij}=-\sin((i-1)(j-1)\pi/N)$, respectively. The sum of weighted error squares is defined as

$$\begin{aligned}J(\boldsymbol{h})&=\sum_{k=0}^{N-1}w(k)|E(k)|^2\\&=\boldsymbol{E}_r'W\boldsymbol{E}_r+\boldsymbol{E}_i'W\boldsymbol{E}_i\\&=\begin{bmatrix}\boldsymbol{E}_r\\ \boldsymbol{E}_i\end{bmatrix}'\begin{bmatrix}W & 0\\ 0 & W\end{bmatrix}\begin{bmatrix}\boldsymbol{E}_r\\ \boldsymbol{E}_i\end{bmatrix},\end{aligned}\tag{5}$$

where $W=\mathrm{diag}[w(0), w(1), \ldots, w(N-1)]$ with $w(k)\geqslant 0$ for all $0\leqslant k\leqslant N-1$. It is well-known that the WLS estimate, $\hat{\boldsymbol{h}}$, of $\boldsymbol{h}$ which minimizes $J(\boldsymbol{h})$ is given by [12]

$$\hat{\boldsymbol{h}}=[D'W_1D]^{-1}D'W_1\begin{bmatrix}\mathrm{Re}(\boldsymbol{H}_d)\\ \mathrm{Im}(\boldsymbol{H}_d)\end{bmatrix}, \tag{6}$$

where $D'=[D_1' \quad D_2']$ and $W_1=\mathrm{diag}[W, W]$. The new approximation method to be presented is based on the following well-known property of WLS estimators.

(P1) *The larger the weight $w(k)$, the smaller is the absolute approximation error $|E(k)|$.*

The new approximation method to be presented is an iterative algorithm based on (P1) for finding the optimum $w(k)$ such that $|E(k)|$ is equiripple with the desired approximation error ratio among the nontransition bands or, equivalently, $|W_e(k)E(k)|$ for $k/2N\in B_{NT}$ is equiripple. Before presenting the new design method, let us define some notations for ease of later use;

- Error ripple $E_m^i(k)$:

$$E_m^i(k)=|E(k)|, \quad k/2N\in B_m^i, \qquad i=1,2,\ldots,q, \tag{7}$$

where $q$ is the total number of error ripples in $B_m$ and

$$B_m^i=\{f\,|\,f_m^{i-1}\leqslant f\leqslant f_m^i\}\subset B_m,$$

where $f_m^0=f_{m1}$, $f_m^q=f_{m2}$ and $f_m^1,\ldots,f_m^{q-1}$ are the frequencies at each of which $|E(f)|$ is a local minimum.

- Amplitude $e_m^i$ of error ripple $E_m^i(k)$:

$$e_m^i=\max\{E_m^i(k), k/2N\in B_m^i\}. \tag{8}$$

- Piecewise-constant function $R(k)$, $k/2N\in B_{NT}$:

$$R(k)=W_e(k)e_m^i=\rho_m e_m^i, \quad \text{if } k/2N\in B_m^i. \tag{9}$$

The new design method, which is shown in Fig. 1, begins with the initial weighting function

$$w^{(0)}(k)=\begin{cases}W_e(k), & k/2N\in B_{NT},\\ 0, & k/2N\in B_{TS}.\end{cases} \tag{10}$$

Assume that we ended up with the weighting function $w(k)=w^{(n-1)}(k)$ at the $(n-1)$th iteration. For the $n$th iteration, the WLS estimate, $\hat{\boldsymbol{h}}$, is computed

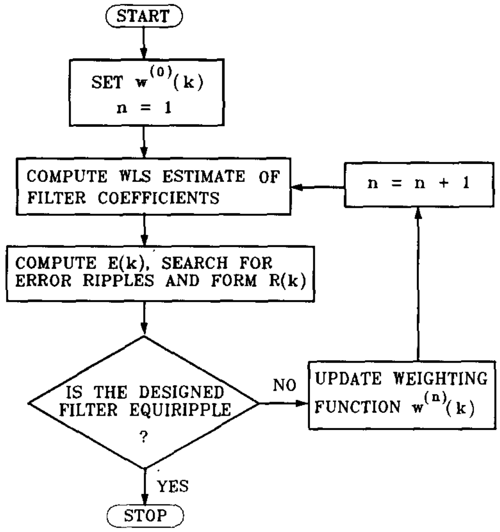


Fig. 1. The proposed WLS Chebyshev approximation method.

by (6), in which $w(k)=w^{(n-1)}(k)$. Then the associated $E(k)$ is computed by (4). Next, we search for $E_m^i(k)$ and $e_m^i$, for all $i$ and $m$, and form $R(k)$. Then we check whether $|W_e(k)E(k)|$ for $k/2N\in B_{NT}$ is equiripple by

$$\frac{R_{\max}-R_{\min}}{R_{\max}}\leqslant\sigma, \tag{11}$$

where

$$R_{\max}=\max\{R(k), k/2N\in B_{NT}\},$$

$$R_{\min}=\min\{R(k), k/2N\in B_{NT}\},$$

and $\sigma$ is a preassigned small positive constant. If $|W_e(k)E(k)|$ is not equiripple yet, we update the weighting function by

$$w^{(n)}(k)=\begin{cases}w^{(n-1)}(k)R(k)/w_{\max}, & k/2N\in B_{NT},\\ 0, & k/2N\in B_{TS},\end{cases} \tag{12}$$

where

$$w_{\max}=\max\{w^{(n-1)}(k)R(k), k/2N\in B_{NT}\} \tag{13}$$

normalizes $w^{(n)}(k)$ such that $0<w^{(n)}(k)\leqslant 1$ for all $k/2N\in B_{NT}$.

It is advisable here to indicate the distinctions between the proposed approximation method and

Chi–Kou's method [3, 4] as well as Chit–Mason's method [5] as follows:

(R1) One can see, from (12) and (9), that the proposed method updates the weighting function $w^{(n)}(k)$ associated with the nontransition bands according to the first power of error ripple amplitudes, whereas Chi–Kou's method updates the weighting function $w^{(n)}(k)$ according to the square of error ripple amplitudes. On the other hand, the weighting function over the transition bands is set to zero by both.

(R2) Chi–Kou's method includes an inner loop and an outer loop. In the inner loop, $w^{(n)}(k)$ is updated only to yield an equiripple frequency response in each frequency band without considering the actual relative approximation error ratio among different frequency bands. After equiripple frequency response has been attained, in the outer loop, $w^{(n)}(k)$ is then adjusted according to the actual approximation error ratio among different frequency bands. On the other hand, the proposed method includes only a single loop (see Fig. 1) within which $w(k)$ is updated according to both amplitudes of error ripples and desired approximation error ratio among different frequency bands simultaneously (see (12) and (9)).

(R3) The convergence of both the proposed approximation method and Chi–Kou's method is supported by the property (P1). By our experience, the proposed approximation method is computationally more efficient than Chi–Kou's method from the viewpoint of number of iterations spent for convergence. Two reasons for this are given as follows. The first one is just the judicious single loop instead of nested loops used by the former as discussed in (R2). The other reason is that the latter often converges towards the optimum $h(n)$ in an over oscillating manner by our experience. Therefore, we update the weighting function in a more conservative fashion as mentioned in (R1) so that the former converges in fewer iterations than the latter.

(R4) One can observe, from (5), that $J(\boldsymbol{h})$ is also the sum of absolute squares of the weighted complex error $\sqrt{w(k)}E(k)$ which can also be expressed as

$$\sqrt{w(k)}E(k)=\sqrt{w(k)}H_{\mathrm{d}}(k)-\sqrt{w(k)}H(k),$$

where $\sqrt{w(k)}H_{\mathrm{d}}(k)$ and $\sqrt{w(k)}H(k)$ are actually the amplitude and phase of the steady-state output of the desired filter as well as the filter under design, respectively, when the common input to both filters is the sinusoidal signal $\sqrt{w(k)}\sin(2\pi(k/2N)n)$. On the other hand, Chit and Mason [5] assume that the common input to both the desired filter and the filter under design is given by

$$\sum_{k=0}^{N-1} C(k)\sin(2\pi(k/2N)n)$$

and that $y_{\mathrm{d}}(n)$ and $y(n)$ are the corresponding outputs of the former and the latter, respectively. The filter coefficients $\boldsymbol{h}$ are obtained by minimizing mean square error of $e(n)=y_{\mathrm{d}}(n)-y(n)$. They iteratively update $C(k)$ and $\boldsymbol{h}$ such that $|W_{\mathrm{e}}(k)E(k)|$ for $k/2N\in B_{\mathrm{NT}}$ is equiripple where $E(k)$ is the discrete Fourier transform of $e(n)$. In other words, conceptually, Chit–Mason's method and the proposed method are based on similar philosophies while the former is a time-domain approach and the latter is a frequency-domain approach.

Although the proposed approximation method is applicable to approximating any arbitrary frequency response, however, some constraints on the filter coefficients $h(n)$ might be implicitly required in some applications. For instance, the linear phase constraint requires either symmetry or antisymmetry with respect to the center of $h(n)$. Therefore, the nonredundant filter coefficients basically consist of only one half of the filter coefficients for this case. Some constraints such as $h(n)=0$ for some $n$'s, symmetry or antisymmetry of $h(n)$ can be easily taken into account in the proposed method by appropriately reformulating the linear model given by (4) such that $\boldsymbol{h}$ only consists of nonredundant filter coefficients.

Finally, let us conclude this section with an analysis below that shows that the following fact (F1) is true.

(F1) *The obtained optimum FIR filter is equiripple with desired approximation error ratio among p nontransition bands.*

From (12) and (13), one can see that

$$w^{(n)}(k)=\frac{w^{(n-1)}(k)R(k)}{\max\{w^{(n-1)}(k)R(k),\ k/2N\in B_{\mathrm{NT}}\}}\quad \forall k/2N\in B_{\mathrm{NT}}. \tag{14}$$

Assume that $w^{(n)}(k)$ converges to $\hat{w}(k)$ as $n$ increases. Thus, we have, from (14), that

$$\hat{w}(k)=\frac{\hat{w}(k)R(k)}{\max\{\hat{w}(k)R(k),\ k/2N\in B_{\mathrm{NT}}\}}\quad \forall k/2N\in B_{\mathrm{NT}}, \tag{15}$$

which implies

$$\begin{aligned} R(k)&=\max\{\hat{w}(k)R(k),\ k/2N\in B_{\mathrm{NT}}\}\\ &=\hat{w}(k')R(k')\quad \forall k/2N\in B_{\mathrm{NT}}, \end{aligned} \tag{16}$$

where $k'$ is associated with the maximum of $\hat{w}(k)R(k)$ for $k/2N\in B_{\mathrm{NT}}$. Letting $k=k'$ in (16), we have

$$\hat{w}(k')=1. \tag{17}$$

Substituting (17) back to (16) yields

$$\begin{aligned} R(k)&=R(k')\\ &=\delta \text{ (constant)}\quad \forall k/2N\in B_{\mathrm{NT}}, \end{aligned} \tag{18}$$

which together with (9) imply that $\rho_m e_m^i=\delta$ for all $i$ and $m$, or $e_m^i=\delta_m=\delta/\rho_m$ for all $i$ (i.e., $|E(k)|$ for $k/2N\in B_m$ is equiripple with maximum error $\delta_m$) as well as $\delta_1:\delta_2:\cdots:\delta_p=(1/\rho_1):(1/\rho_2):\cdots:(1/\rho_p)$.

## 3. Design examples

In order to demonstrate that the proposed WLS Chebyshev approximation method works well, five design examples including a linear phase lowpass filter, a nonlinear phase lowpass filter, a full-band differentiator, a chirp all-pass phase equalizer and a sine-delay all-pass phase equalizer using the proposed method with the parameter $\sigma$ set to 0.01 are presented in the following, respectively.

*EXAMPLE 1.* Linear phase lowpass filter (taken from [3, 4, 13]).

The desired frequency response $H_{\mathrm{d}}(f)$ is given as follows:

$$H_{\mathrm{d}}(f)=\begin{cases} e^{-j2\pi f(M-1)/2}, & \text{if } f\in B_1=[0,0.2] \text{ (passband)},\\ 0, & \text{if } f\in B_2=[0.3,0.5] \text{ (stopband)}. \end{cases}$$

The filter to be designed is a 2-band ($p=2$) linear phase FIR filter of length $M=28$ (type II) and

$$W_{\mathrm{e}}(f)=\begin{cases} \rho_1=0.1, & f\in B_1,\\ \rho_2=1, & f\in B_2. \end{cases}$$

In this example, $N=2000$ was used. For a type II linear phase FIR filter, the filter coefficients must satisfy the constraint

$$h(n)=h(M-1-n),$$

which, as mentioned previously, can be taken into account in the linear model (4) with $\boldsymbol{h}=[h(0),h(1),\ldots,h(M/2-1)]'$ instead. The total iterations spent by the proposed approximation method was 10, which is much smaller than 39, the total iterations spent by Chi–Kou's method reported in [3, 4]. Figure 2(a) shows the magnitude response of the designed lowpass linear phase FIR filter. Figure 2(b) shows the absolute approximation error $|E(f)|$ which is equiripple with maximum errors $\delta_1=0.0092$ and $\delta_2=0.00092$ in the passband and the stopband, respectively. Note that $\delta_1/\delta_2\approx(1/\rho_1)/(1/\rho_2)=10$. These results are the same as the corresponding results reported in [3, 4] obtained by Chi–Kou's method as well as those reported in [18] obtained by Parks–McClellan's method for the same example.

*EXAMPLE 2.* Lowpass filter with constant group delay $\tau_{\mathrm{d}}$ (taken from [2, 15, 16, 21]).

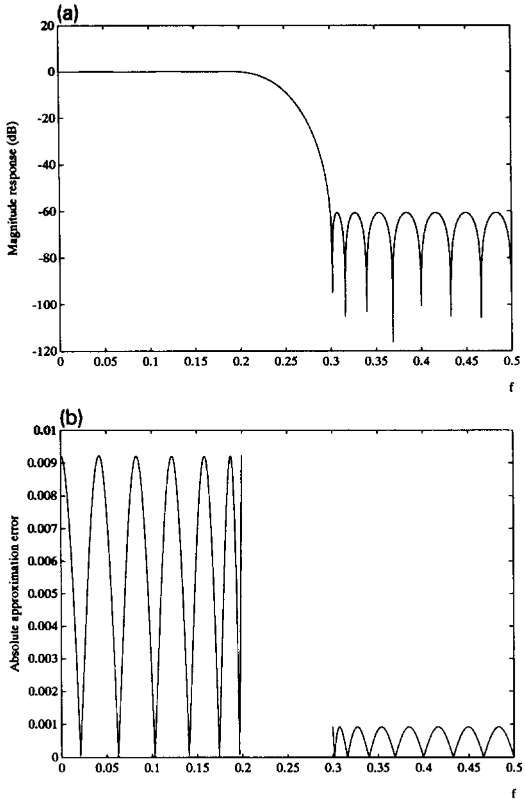


Fig. 2. Example 1. The designed linear phase lowpass filter with $M=28$ (type II), passband $=[0, 0.2]$ and stopband $=[0.3, 0.5]$. (a) Magnitude response, (b) absolute approximation error.

The desired frequency response $H_d(f)$ is given as follows:

$$H_d(f) = \begin{cases} e^{-j2\pi\tau_d f}, & \text{if } f \in B_1 = [0, 0.06] \text{ (passband)}, \\ 0, & \text{if } f \in B_2 = [0.12, 0.5] \text{ (stopband)}, \end{cases}$$

where $\tau_d = 12$. The filter to be designed is a 2-band filter ($p=2$) of length $M=31$ and

$$W_e(f) = \begin{cases} \rho_1 = 0.1, & f \in B_1, \\ \rho_2 = 1, & f \in B_2. \end{cases}$$

In this example, $N=1000$ was used. The total iterations spent by the proposed approximation method was 11. The magnitude response and group delay response of the designed filter are shown in Figs. 3(a) and 3(b), respectively. One can see, from Fig. 3(b), that in the passband the group delay is nearly constant although it oscillates around 12 with a maximum deviation of 1.096. Figure 3(c) shows

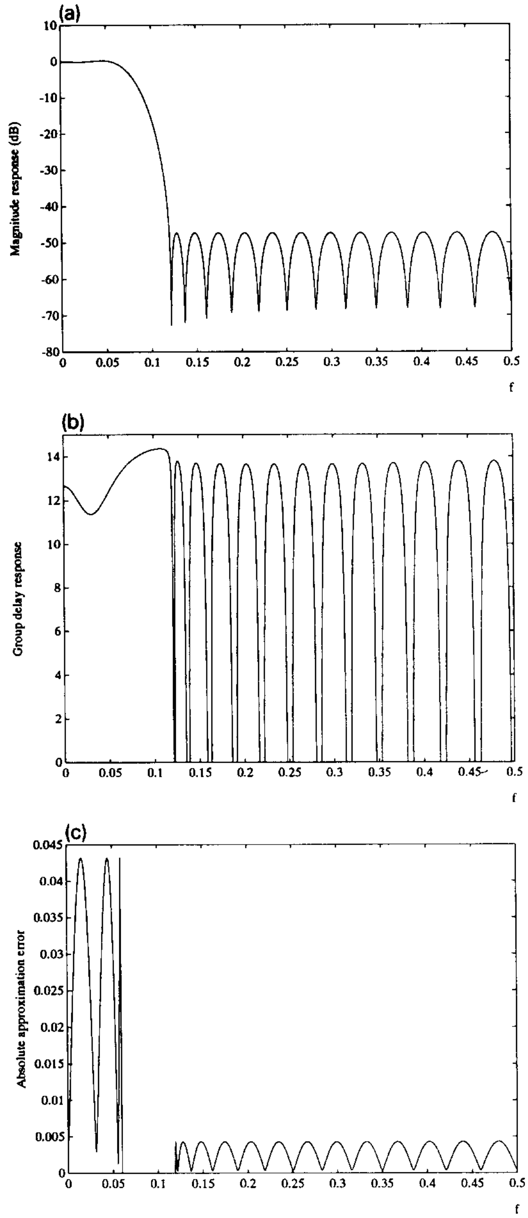


Fig. 3. Example 2. The designed lowpass filter with $M=31$, $\tau_d=12$, passband $=[0, 0.06]$ and stopband $=[0.12, 0.5]$. (a) Magnitude response, (b) group delay response, (c) absolute approximation error.

the absolute approximation error $|E(f)|$ which is equiripple with maximum errors $\delta_1=0.0441$ and $\delta_2=0.00443$, in the passband and the stopband, respectively. Note that, again, $\delta_1/\delta_2\approx(1/\rho_1)/(1/\rho_2)=10$. These results together with the corresponding results for the same example reported in [2, 15, 16, 21] are shown in Table 1, from which one can see that they are all comparable to one another.

*EXAMPLE 3.* Full-band differentiator with constant group delay $\tau_d$ (taken from [15]).

The desired frequency response $H_d(f)$ is given as follows:

$$H_d(f)=\mathrm{j}2\pi f\,\mathrm{e}^{-\mathrm{j}2\pi\tau_d f},\quad 0\leqslant f\leqslant 0.5,$$

where $\tau_d=11.5$. The filter to be designed is a single band ($p=1$) filter with length $M=31$ and

$$W_e(f)=1,\quad 0\leqslant f\leqslant 0.5.$$

In this example, $N=1000$ was used. The total iterations spent by the proposed approximation method was also 11. Figures 4(a) and 4(b) show the magnitude response and group delay response, respectively. One can see, from Fig. 4(b), that there is a large group delay error at the origin ($f=0$) due to an unavoidable phase discontinuity at the origin. Since the magnitude response at the origin is zero, the group delay error at the origin is of no consequence. Figure 4(c) shows $|E(f)|$ which, again, is equiripple with maximum error 0.0185 which is slightly smaller than the corresponding maximum error 0.01952 obtained by Pei–Shyu's method reported in [15].

*EXAMPLE 4.* Chirp all-pass phase equalizer with linear group delay (taken from [22]).

The desired frequency response $H_d(f)$ is given as follows:

$$H_d(f)=\mathrm{e}^{-\mathrm{j}[2\pi fL+\beta(2\pi f-\pi/2)^2]},\quad 0\leqslant f\leqslant 0.5,$$

where $L=(M-1)/2$ and $\beta=16/2\pi$. The desired group delay is $L+2\beta(2\pi f-\pi/2)$. The filter to be designed is a single band filter ($p=1$) of length $M=61$ and

$$W_e(f)=1,\quad 0\leqslant f\leqslant 0.5.$$

In this example, $N=1000$ was used. Because the phase $-\beta(2\pi f-\pi/2)^2$ of the all-pass filter $\mathrm{e}^{\mathrm{j}2\pi fL}\cdot H_d(f)$ is symmetric with respect to $f=0.25$, the filter coefficients $h(n)$ must satisfy the following constraints [22]:

$$h(L-n)=h(L+n),\quad \text{if } n \text{ is even},$$
$$h(L-n)=-h(L+n),\quad \text{if } n \text{ is odd}.$$

Remark that these constraints can be easily put into (4) and the resulting $\boldsymbol{h}$ consists only of non-redundant filter coefficients. The total iterations spent by the proposed approximation method was 10. The magnitude response and absolute approximation error $|E(f)|$ of the designed filter are shown in Figs. 5(a) and 5(b), respectively. One can see, from Fig. 5(b), that the absolute approximation error $|E(f)|$ is equiripple with maximum error $\delta=0.00107$ which is smaller than the corresponding maximum errors 0.001595 and 0.02 (read from Fig. 5 in [5]) obtained by Pei–Shyu's method [15] and Chit–Mason's method [5], respectively. Figures 5(c) and 5(d) show the group delay response and

Table 1

Numerical results of Example 2 (lowpass filter with constant group delay). Maximum approximation errors $\delta_1$ (passband), $\delta_2$ (stopband) and maximum deviation of group delay (passband) obtained by the proposed method and the corresponding results reported in [2, 15, 16, 21]. $M=31$, $\tau_d=12$, $B_1=[0, 0.06]$ (passband) and $B_2=[0.12, 0.5]$ (stopband)

| | Proposed method | Chen–Parks method [2] | Preuss method [16] | Schulist method [21] | Pei–Shyu method [15] |
|---|---|---|---|---|---|
| $\delta_1$ | 0.0441 | 0.0436 | 0.0426 | 0.0425 | 0.04404 |
| $\delta_2$ | 0.00443 | 0.00436 | 0.00426 | 0.00425 | 0.004401 |
| Maximum deviation of group delay in passband | 1.096 | 0.97 | 1.111 | – | 1.063 |

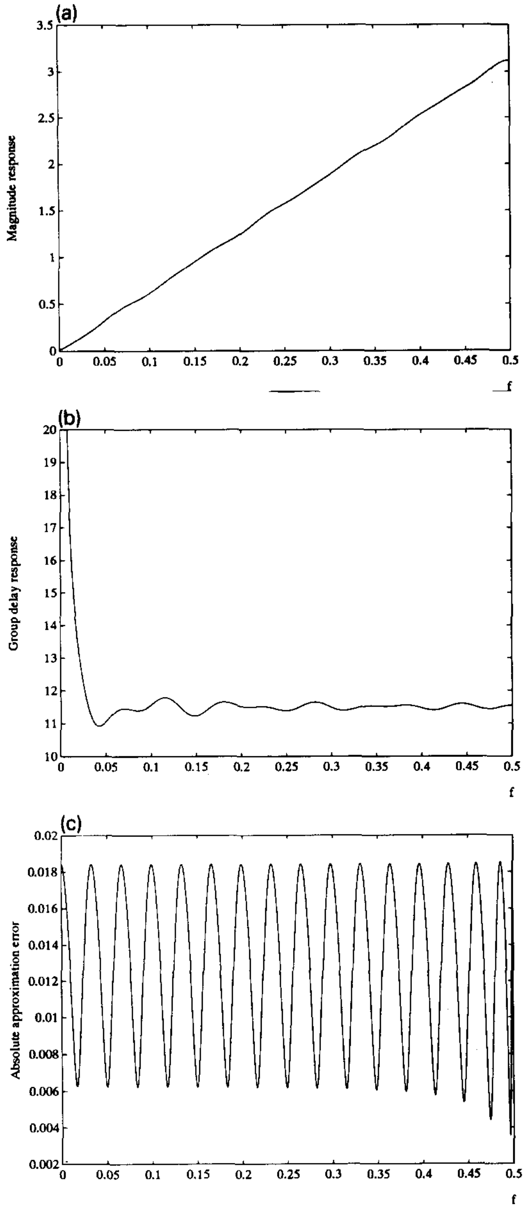


Fig. 4. Example 3. The designed full-band differentiator with $M=31$ and $\tau_d=11.5$. (a) Magnitude response, (b) group delay response, (c) absolute approximation error.

the group delay error response, respectively. One can see, from Fig. 5(d), that maximum deviation of group delay occurs at both zero and folding frequencies but its value is only 0.0926 which is also smaller than Pei–Shyu's group delay error 0.1146 reported in [15].

*EXAMPLE 5.* Sine-delay all-pass phase equalizer (taken from [22]).

The desired frequency response $H_d(f)$ is given as follows:

$$H_d(f)=e^{-j[2\pi fL-2\pi(1-\cos(2\pi f))]},\quad 0\leqslant f\leqslant 0.5,$$

where $L=(M-1)/2$. The desired group delay is $L-2\pi\sin(2\pi f)$. The filter to be designed is also a single band filter ($p=1$) of length $M=61$ and

$$W_e(f)=1,\quad 0\leqslant f\leqslant 0.5.$$

In this example, $N=1000$ was used. Because the phase $2\pi(1-\cos(2\pi f))$ of the all-pass filter $e^{j2\pi fL}\cdot H_d(f)$ is antisymmetric with respect to $f=0.25$, the filter coefficients $h(n)$ must satisfy the following constraint [22]:

$$h(L-n)=h(L+n)=0,\quad \text{if } n \text{ is odd.}$$

Again, we put this constraint into (4) appropriately to keep the antisymmetry property of the desired phase. Then $\boldsymbol{h}=[h(0),h(2),\ldots,h(M-1)]'$ for this case. The total iterations spent by the proposed approximation method was also 10. The magnitude response and absolute approximation error $|E(f)|$ of the designed filter are shown in Figs. 6(a) and 6(b), respectively. One can see, from Fig. 6(b), that the absolute approximation error is equiripple with maximum error $\delta=0.00097$ which is comparable with Chit–Mason's maximum error 0.00094 (read from Fig. 4 in [5]) and meanwhile is smaller than Pei–Shyu's maximum error 0.001528 [15] for the same example. Figures 6(c) and 6(d) show the group delay response and the group delay error response, respectively. One can see, from Fig. 6(d), that maximum deviation of group delay occurs at both zero and folding frequencies but its value is only 0.1015 which is also smaller than Pei–Shyu's group delay error 0.1258 reported in [15].

The previous design examples demonstrate that the proposed WLS Chebyshev approximation method works well, and the number of iterations spent in each example is also small (around 10). As stated in (F1), the designed filters are indeed equiripple with desired approximation error ratio among nontransition bands. We also performed

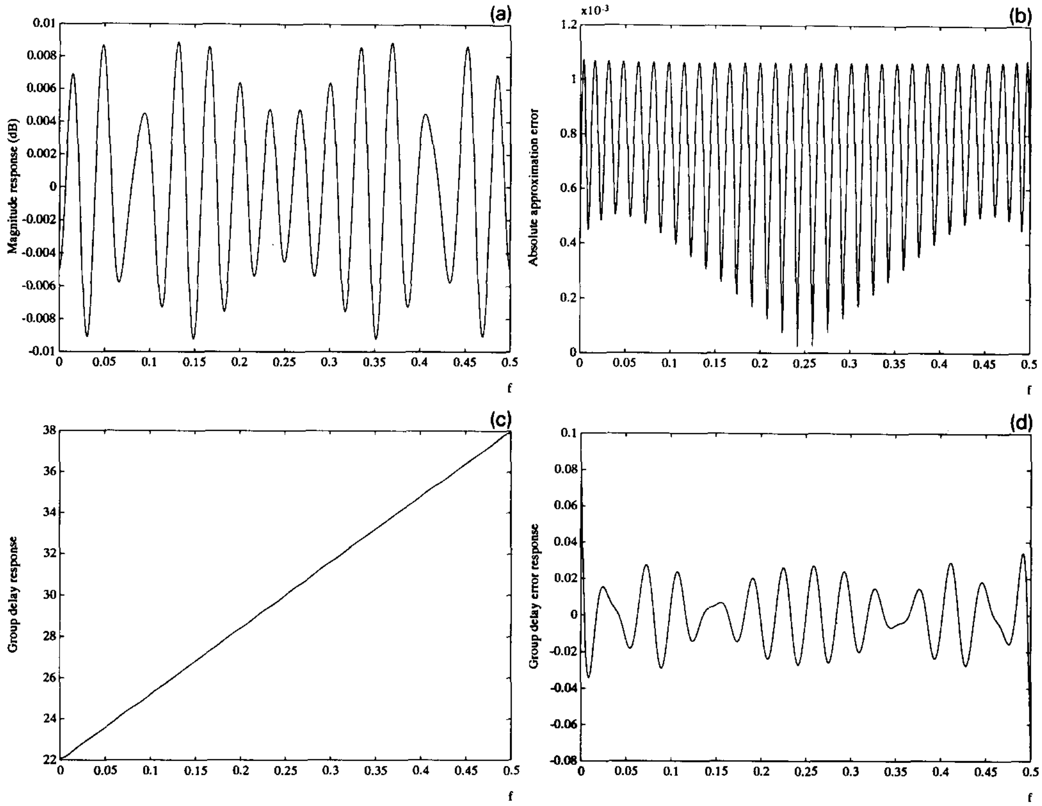


Fig. 5. Example 4. The designed chirp all-pass phase equalizer with $M = 61$. (a) Magnitude response, (b) absolute approximation error, (c) group delay response, (d) group delay error response.

the designs described in Examples 1, 4 and 5 using the proposed approximation method without taking into account of the implicit constraints on filter coefficients. However, the obtained results are almost indistinguishable from the previous results for each example. The reason for this is that the value for parameter $N$ used in each of these three examples is large enough to reflect the implicit constraints on the designed filter. On the other hand, the previous examples also support that the proposed WLS Chebyshev approximation method has better performance than both Chit–Mason's method and Pei–Shyu's method.

## 4. Discussion and conclusions

We have presented a new self-initiated iterative WLS approximation method (see Fig. 1) for the design of FIR filters with arbitrary complex frequency response and real filter coefficients. Both the proposed method and Chi–Kou's method are based on the property (P1) of the well-known WLS estimator although the latter is only applicable to the design of FIR filters with linear phase. Contrast to Chi–Kou's method, the proposed method adjusts the filter coefficients by simultaneously considering both approximation error in each frequency band and desired relative approximation error ratio among all frequency bands of interest (see (R2)). The key weighting function $w(k) = w^{(n)}(k)$ (see (12)) over the nontransition bands obtained in the $n$th iteration is determined by error ripple amplitudes instead of squares of error ripple amplitudes as in Chi–Kou's method (see (R1)), while the part of $w(k)$ associated with the transition bands is set to zero. We also showed five

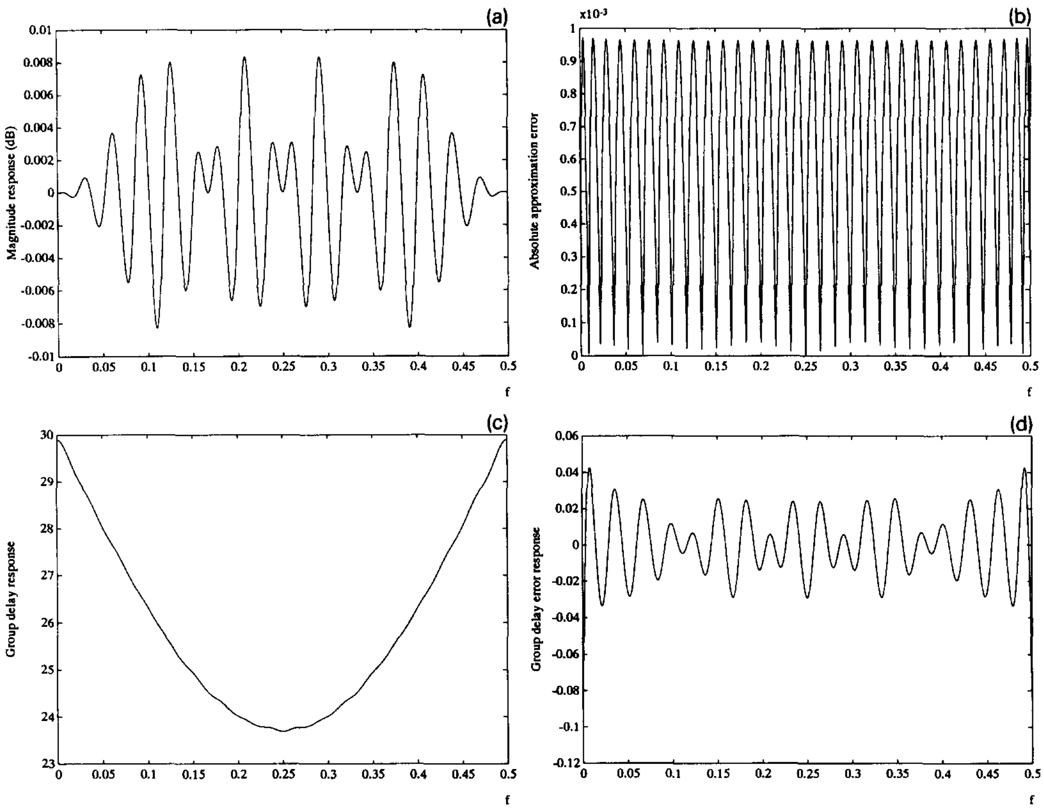


Fig. 6. Example 5. The designed sine-delay all-pass phase equalizer with $M=61$. (a) Magnitude response, (b) absolute approximation error, (c) group delay response, (d) group delay error response.

design examples to justify that the proposed approximation method works well and support that its performance is superior to both Pei–Shyu's method and Chit–Mason's method. Finally, let us summarize some other remarks with regard to the proposed approximation method as follows:

(1) The proposed method is computationally more efficient than the Chi–Kou's method for the linear phase case from the viewpoint of total iterations spent for convergence by our experience (see (R3) and Example 1).

(2) As Chi–Kou's method, the proposed design method does not need any initial guess for a suitable set of extremal frequencies or filter coefficients.

(3) As both Chi–Kou's method and Parks–McClellan's method, the proposed approximation method also allows the exact specification of the cutoff frequencies because the cutoff frequencies determine the frequency interval $B_{NT}$ formed by the nontransition bands over which the whole design procedure is to make $|W_e(k)E(k)|$ equiripple.

(4) If the implicit constraints on filter coefficients can be obtained ahead of time, as mentioned previously, they can be easily taken into account in the proposed approximation method (also see Examples 1, 4 and 5). If they are not known and thus are ignored by the proposed approximation method, the same optimum filter can still be obtained as long as the value for parameter $N$ (number of uniform samples of the desired frequency response used for the linear model (4)) used is large enough.

(5) Contrast to Chit–Mason's time-domain LMS approximation methods [5, 10], the proposed

method is a frequency-domain WLS method based on similar philosophies (see (R4)). However, the latter has better performance than the former although the designed optimum filters by both methods are equiripple.

(6) The proposed method is also computationally efficient since the direct solution for the optimum filter coefficients by the WLS estimator (see (6)) is quite straightforward. The WLS estimate can also be efficiently computed in a recursive fashion without matrix inversion [12].

(7) The proposed algorithm is also applicable in the case of complex filter coefficients by replacing the linear model (4) and the WLS estimate (6) with

$$\boldsymbol{E} = \boldsymbol{H}_{\mathrm{d}} - \boldsymbol{D}\boldsymbol{h} \tag{19}$$

and

$$\hat{\boldsymbol{h}} = [D^{\mathrm{H}} W D]^{-1} D^{\mathrm{H}} W \boldsymbol{H}_{\mathrm{d}}, \tag{20}$$

respectively, where $E(k)$ in $\boldsymbol{E}$ denotes $E(f=k/N)$, $H_{\mathrm{d}}(k)$ in $\boldsymbol{H}_{\mathrm{d}}$ denotes $H_{\mathrm{d}}(f=k/N)$, $D^{\mathrm{H}}$ is the complex conjugate transpose of $N \times M$ matrix $D$ whose $(k, i)$th element is equal to $\exp\{-\mathrm{j}2\pi(k-1)(i-1)/N\}$. Note that $N$ is the total number of uniform samples in the interval [0, 1] for this case.

## Acknowledgments

The research described in this paper was performed at the Department of Electrical Engineering, National Tsing Hua University, Hsinchu, Taiwan, ROC, and was supported by the National Science Council under Grant NSC81-0404-E-007-001.

## References

[1] I. Barrodale, L.M. Delves and J.C. Mason, "Linear Chebyshev approximation of complex-valued functions", *Math. Comp.*, Vol. 32, 1978, pp. 853–863.

[2] X. Chen and T.W. Parks, "Design of FIR filters in the complex domain", *IEEE Trans. Acoust. Speech Signal Process.*, Vol. ASSP-35, February 1987, pp. 144–153.

[3] C.-Y. Chi and Y.-T. Kou, "A new self-initiated optimum WLS approximation method for the design of linear phase FIR digital filters", *Proc. IEEE 1991 Internat. Symp. Circuits and Systems*, Singapore, June 1991, Vol. 1, pp. 168–171.

[4] C.-Y. Chi and Y.-T. Kou, "A new self-initiated optimum WLS Chebyshev approximation method for the design of linear phase FIR digital filters", *IEEE Trans. Circuits and Systems*, submitted.

[5] N.N. Chit and J.S. Mason, "Complex Chebyshev approximation for FIR digital filters", *IEEE Trans. Signal Process.*, Vol. 39, January 1991, pp. 49–54.

[6] G. Cortelazzo and M.R. Lightner, "Simultaneous design in both magnitude and delay of IIR and FIR filters based on multiple criterion optimization", *IEEE Trans. Acoust. Speech Signal Process.*, Vol. ASSP-32, 1984, pp. 949–967.

[7] S. Ellacott and J. Williams, "Linear Chebyshev approximation in the complex plane using Lawson's algorithm", *Math. Comp.*, Vol. 30, 1976, pp. 35–44.

[8] K. Glashoff and K. Roleff, "A new method for Chebyshev approximation of complex-valued functions", *Math. Comp.*, Vol. 36, 1981, pp. 233–239.

[9] A.T. Johnson, Jr., "Simultaneous magnitude and phase equalization using digital filters", *IEEE Trans. Circuits and Systems*, Vol. CAS-25, May 1978, pp. 319–321.

[10] J.S. Mason and N.N. Chit, "New approach to the design of FIR digital filters", *IEE Proc.*, Pt. G, Vol. 134, August 1987, pp. 167–180.

[11] J.H. McClellan and T.W. Parks, "A unified approach to the design of optimum FIR linear phase filters", *IEEE Trans. Circuit Theory*, Vol. CT-20, November 1973, pp. 697–701.

[12] J.M. Mendel, *Lessons in Digital Estimation Theory*, Prentice Hall, Englewood Cliffs, NJ, 1987.

[13] A. Oppenheim and R. Schafer, *Discrete-Time Signal Processing*, Prentice Hall, Englewood Cliffs, NJ, 1979.

[14] T.W. Parks and J.H. McClellan, "Chebyshev approximation for nonrecursive digital filters with linear phase", *IEEE Trans. Circuit Theory*, Vol. CT-19, March 1972, pp. 189–194.

[15] S.-C. Pei and J.-J. Shyu, "Design of real FIR filters with arbitrary complex frequency response by two real Chebyshev approximations", *Signal Processing*, Vol. 26, No. 1, January 1992, pp. 119–129.

[16] K. Preuss, "On the design of FIR filters by complex Chebyshev approximation", *IEEE Trans. Acoust. Speech Signal Process.*, Vol. ASSP-37, May 1989, pp. 702–712.

[17] J.R. Rice, *The Approximation of Function*, Addison-Wesley, Reading, MA, 1969.

[18] J.R. Rice and K.H. Usow, "The Lawson algorithm and extensions", *Math. Comp.*, Vol. 22, 1968, pp. 118–127.

[19] T.J. Rivlin, *The Chebyshev Polynomials*, Wiley, New York, 1974.

[20] T.J. Rivlin and H.S. Shapiro, "A unified approach to certain problems of approximation and minimization", *J. Soc. Indust. Appl. Math.*, Vol. 9, 1961, pp. 670–699.

[21] M. Schulist, "Improvements of a complex FIR filter design algorithm", *Signal Processing*, Vol. 20, No. 1, May 1990, pp. 81–90.

[22] K. Steiglitz, "Design of digital phase networks", *IEEE Trans. Acoust. Speech Signal Process.*, Vol. ASSP-29, 1981, pp. 171–176.

[23] R.L. Streit and A.H. Nuttall, "A general Chebyshev complex function approximation procedure and an application to Beamforming", *J. Acoust. Soc. Amer.*, Vol. 72, 1982, pp. 181–190.

[24] R.L. Streit and A.H. Nuttall, "A note on the semi-infinite programming approach to complex approximation", *Math. Comp.*, Vol. 40, 1983, pp. 599–605.

[25] L.N. Trefethen, "Near circularity of the error curve in complex Chebyshev approximation", *J. Approx. Theory*, Vol. 31, 1981, pp. 344–367.